Developing Mathematical Thinking

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Abstract: *This Contest Corner Column discusses a generalized problem that provides a challenge to most students and reinforces concepts useful to developing the kind of thinking needed to excel in mathematics competition.*

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The vertices of $\triangle PQR$ have coordinates at $P(0, a)$, $Q(b, 0)$, and $R(c, d)$. If a, b, c, and d are all positive numbers, find the area of $\triangle PQR$ if $c > b$.

This problem is an excellent example of a generalized problem that provides a challenge to most students and reinforces concepts useful to developing the kind of thinking needed to excel in mathematics competition. Most students find it easier to solve specific problems and one might start by asking a class or math club to first try the problem below while knowing that the general solution is the ultimate goal.

The vertices of $\triangle PQR$ have coordinates at $P(0, 2)$, $Q(5, 0)$, and $R(7, 8)$. Find the area of $\triangle PQR$. ***Answer***: 22 sq. units

Having students work collaboratively in small groups of two or three can enhance the learning experience and build problem solving skills. This allows them to see each other's perspectives and ways of thinking about problem solving. Once students have solved the specific problem they will find it easier to solve the general problem because they have a logical series of steps and a plan of attack. Here are two possible solutions to the general problem.

A ***typical solution*** to this problem is to start by drawing the problem and adding line segment RS which is perpendicular to the x-axis (see Figure 1).

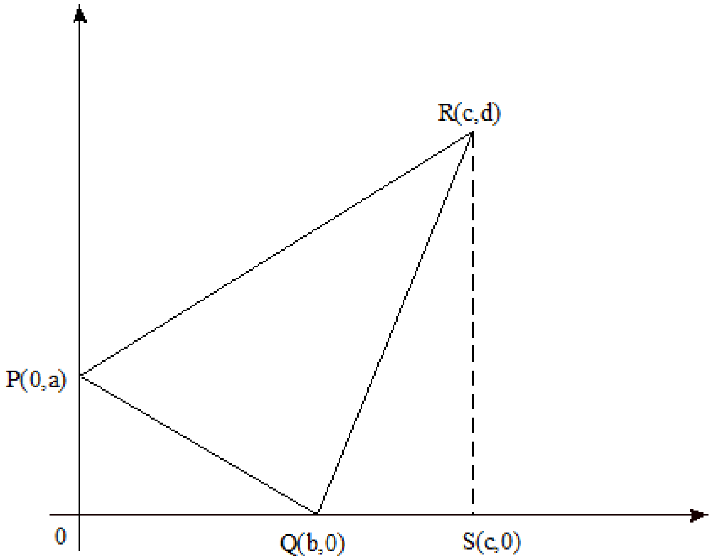

Fig. 1: *Typical solution.*

Note that the area of $\triangle PQR$ = Area of Trapezoid OPRS – area $\triangle OPQ$–$\triangle SQR$. Since the area of a trapezoid is $\frac{1}{2}$ `(sum of the bases)` $\times$ `(height)` and the area of a triangle is $\frac{1}{2}$ `(base)` $\times$ `(height)`, we have area of $\triangle PQR = \frac{1}{2}(a+d)(c) - \frac{1}{2}(b)(a) - \frac{1}{2}(c-b)(d)$ which gives $\triangle PQR = \frac{1}{2}(bd–ab+ac)$.

A more advanced solution to this problem provides an excellent opportunity to introduce or review the concept of determinants and vectors. Determinants and vectors are extremely useful in both geometry and algebra and their mastery can be a powerful tool in competition. Figure 2 shows how vectors u and v define a parallelogram.

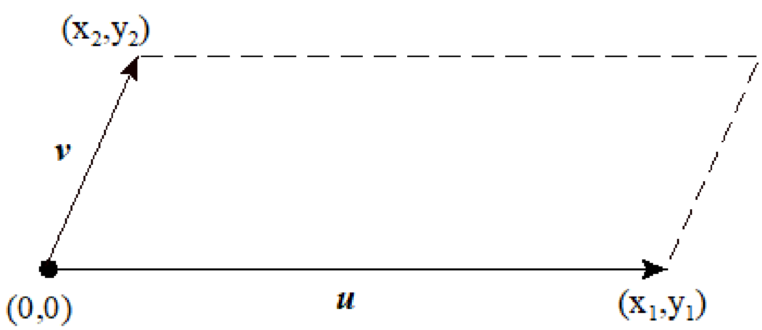

Fig. 2: *Vectors u and v define a parallelogram.*

Recall that the area of a parallelogram in the xy coordinate plane with sides determined by vectors $u = (x_1, y_1)$ and $v = (x_2, y_2)$ is the absolute value of the determinant:

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Therefore, the area of $\triangle PQR = \frac{1}{2}$(Area of Parallelogram $PQSR$). Figure 3 shows this relationship.

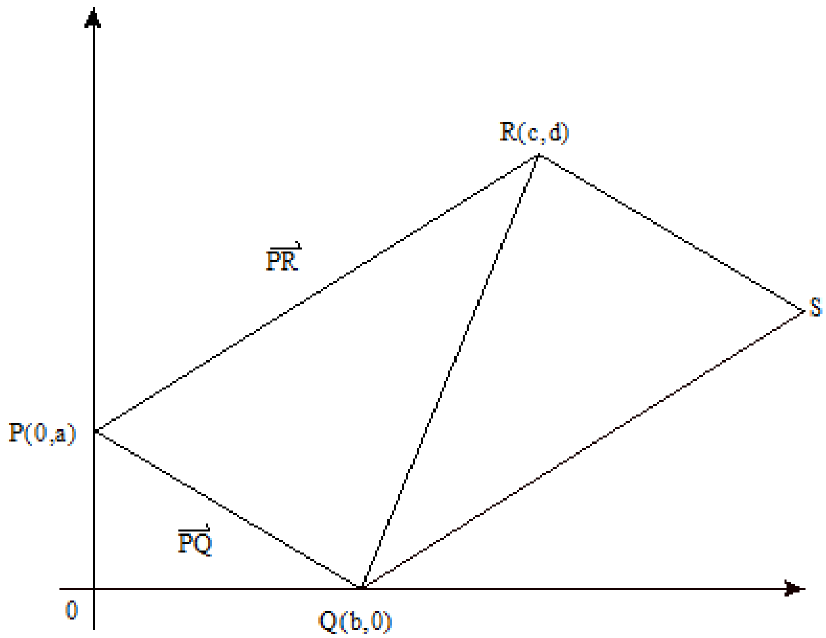

Fig. 3: *Relationship between $\triangle PQR$ and parallelogram $PQSR$.*

Since vector PR is $R–P = (c, d)–(0, a) = (c, d - a)$ and vector $PQ = Q–P = (b, 0)–(0, a) = (b, -a)$, we have an ***alternative solution:***

$$Area = \frac{1}{2}\begin{vmatrix} b & -a \\ c & (d-a) \end{vmatrix} = \frac{1}{2}(bd - ab + ac)$$

So, a theorem stating what we have discovered might read as follows:

> If the vertices of $\triangle PQR$ have coordinates at $P(0, a)$, $Q(b, 0)$, and $R(c, d)$ with a, b, c, and d being positive numbers with $c > b$, then the area of $\triangle PQR = \frac{1}{2}(bd–ab + ac)$.

Try this problem with a class or math club. Let teams come up with solutions and present their results. You will be surprised at how many different solutions and approaches students can find. If no one comes up with the determinant approach, the teacher can make that presentation. Our Fall 2005 Contest Corner discussed differentiation in mathematics instruction and the above problem was used as an example.

Be sure to visit the State Tournament of Mathematics web page `octmtournament.org` to find important dates related to the annual competition. You will also find copies of competition problems dating from 2004 to present with solutions that can inspire your mathletes and further to develop their problem solving skills.

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