Getting Past the Sticking Points: A Questioning Framework for Fraction Multiplication

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Abstract: *While the procedure for multiplying a fraction times a fraction may seem straightforward, understanding why the procedure works is complex. Ohio's New Learning Standards for mathematics identify the need to use visual models and story contexts to develop student understanding of fraction multiplication. This article outlines a questioning framework that, when used with visual models and story contexts, can help students with four common fraction $\times$ fraction "sticking" points.*

Keywords: *Number Concepts and Operations, Rational Number, Fraction Multiplication*

1 Introduction

As procedures go, fraction $\times$ fraction multiplication seems straightforward. Just multiply the denominator of each fraction to get the denominator in the answer and then multiply the numerator of each fraction to get the numerator in the answer. However, knowledge of the algorithm alone does little to support the development of students' number sense or fraction operation sense. Students develop a better conceptual understanding of fraction quantities if fraction $\times$ fraction multiplication is explored using problem contexts and visual diagrams. This approach is supported in Ohio's New Learning Standards for fifth grade, Number and Operations—Fractions 5.NF.4 and 5.NF.5, which state that products should be interpreted as parts of a partition, that visual fraction models should be used when teaching fraction multiplication, and that students should be able to create contexts for fraction multiplication.

Such an approach means that teachers need to engage students in productive mathematical discussions about how they are solving contextual problems and what their visual diagrams represent. One challenge a teacher faces when engaging students in productive mathematical discussions of their work is how to support learners without taking over and reducing the level of mathematical work for students. As I work with teachers, many note that they struggle to guide students when they face mathematical obstacles. Too often, teachers find themselves explaining what to do rather than redirecting students in a way that helps them think and reason. Through work with these teachers, I have developed a questioning framework to support discussions about fraction $\times$ fraction multiplication in contextual situations. In this article, I examine fraction $\times$ fraction multiplication using this framework within the context of buying brownie pans (adapted from Lappan et. al, 2009). I examine four "sticking points" or common areas of struggle that arise when students explore fraction $\times$ fraction multiplication in a contextual situation. The questioning framework can be used to move discussions forward while maintaining a conceptual underpinning of the algorithm.

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2 Exploring Fraction × Fraction Multiplication Through Partitioning

Teachers can introduce students to fraction × fraction multiplication using contexts that involve taking a part of a part of a whole. Armstrong and Bezuk (1995) argue that partitioning experiences can help students make sense of the part and the whole when multiplying fractions. Important ideas for students to grasp include the notion of fractions as scale factors (in addition to quantities) and recognition that the unit can change during the course of one problem. Consider the following situation:

> At the bake shop, there is a pan of brownies in the display case. The pan is 1/2 full. Ms. Jones asks to buy 3/4 of what is in the brownie pan. How much of the pan of brownies will Ms. Jones buy?

This situation is represented as finding 3/4 of 1/2 of the pan of brownies or 3/4 × 1/2. The 1/2 is a quantity. It is the portion of the pan with brownies in it. The fraction 3/4 is a scale factor. It is an action—you are finding 3/4 of 1/2 of a pan of brownies.

In this situation, where someone wants to buy 3/4 of 1/2 of a pan of brownies, multiple levels of partitioning take place. Initially, as expressed in Figure 1a, the whole pan of brownies is partitioned into two parts so that half can be represented. Next, Figure 1b illustrates that 1/2 of a pan is partitioned into four parts so that three "fourths" of the 1/2 of the pan can be identified. Finally, as shown in Figure 1c, the whole pan is partitioned so that the size of the three "fourths" relative to the whole pan (i.e., the unit) can be determined. Knowing this allows the student to determine what fractional part of the whole pan Ms. Jones will buy.

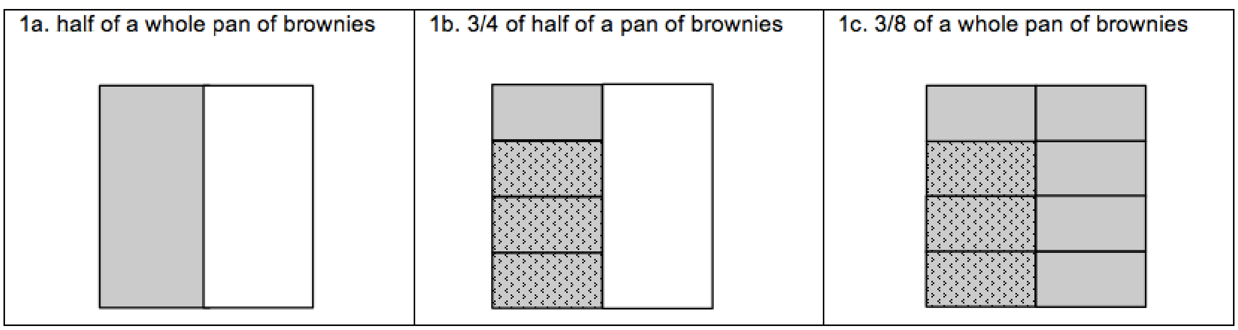


Fig. 1: *Partitioning a pan of brownies to figure out how much 3/4 of 1/2 of the whole pan is.*

In the brownie pan scenario a shifting of the unit takes place. Initially, the unit is the whole pan. The problem starts with 1/2 of the whole pan of brownies. Next, 1/2 of the pan is partitioned into fourths so that 3/4 of 1/2 can be identified. Now half of the pan is the unit we are finding 3/4 of. Finally, the unit shifts back to the whole pan when finding how much 3/4 of 1/2 of a whole pan is. The final answer is expressed in eighths because the whole pan is partitioned into eighths.

3 Helping Students Visualize what is Happening in the Brownie Pan Situation

While it might be tempting to provide students with the shortcut that "of" means "to multiply," enacting multiplication with fractional numbers is much more complex. Students can make sense of this complexity by modeling contextual problems that lead to multiplication. As students work through problem scenarios, the teacher can ask questions to support student sense-making. Figure

2 shows a questioning framework for teachers to help students make sense of fraction × fraction problems. The first two questions are general questions to ask when the teacher first starts working with a student. These questions help assess initial student conceptual understanding. Knowing what students are thinking helps the teacher formulate follow-up questions.

1. What is the problem asking you to do?
2. Tell me about your picture. What does it show?
3. How much are you starting with? How can you show that?
4. What is the problem asking you to find? How can you show that?
5. How much of what you started with do you need?
6. How much of the whole pan do you need?
7. How many pieces are in a whole pan?
8. How many of those pieces do you end up needing?
9. Do you have more than you started with or less? Why does that make sense?
10. What number sentence would you write to show what the problem is asking you to do mathematically?

Fig. 2: *Questioning framework for fraction × fraction multiplication.*

Figure 3 shows how certain questions in the framework can be used with different aspects of the brownie pan problem. Rather than show students what to do, a teacher can pose questions to focus attention on particular mathematical ideas while students work to develop a picture that captures what is happening in the brownie pan scenario. Figure 3a provides a model of what students might draw when solving the brownie pan problem. Question 3 in Figure 3a ("How much are you starting with? How can you show that?") can be used to establish which fraction is the starting quantity. In this case, we start with a pan of brownies that is 1/2 full. In the brownie pan context, 1/2 of a pan is the quantity. The picture in Figure 3a shows a pan that is half full of brownies. Question 4 ("What is the problem asking you to find? How can you show that?") focuses on the role of the scale factor 3/4.

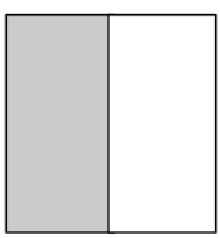
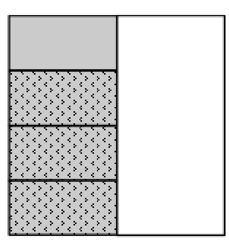
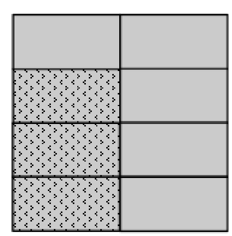
3a. half of a whole pan of brownies	3b. 3/4 of half of a pan of brownies	3c. 3/8 of a whole pan of brownies
		
Q3: How much are you starting with? [1/2 of a pan of brownies]	Q4: What is the problem asking you to find? Q5: How much of what you started with do you need? [3/4 of 1/2 of a pan of brownies]	Q6: How much of a whole pan do you need? Q7: How many pieces are in the whole pan? Q8: How many of those pieces do you end up needing? [3/8 of a whole pan of brownies; 8; 3]

Fig. 3: *Model with Corresponding Questions for Representing 3/4 of 1/2 of a Pan of Brownies.*

3.1 Sticking Point 1: What fraction do I start with?

Asking questions 1 through 4 helps with a common sticking point—which fraction students start with and which fraction students multiply by when modeling fraction multiplication. Often, students want to start with the first fraction they encounter when reading a problem. Questions 3 and 4 direct reasoning toward what each of the fractions in the multiplication problem represent visually. While a teacher could tell students which fraction they should draw first when making their model, the expectation is for students to understand what each fraction represents—that one is a quantity and one is a scale factor. Questions that ask students to read (and to reread) the problem support their ability to process and reason about what the problem is describing and asking.

3.2 Sticking Point 2: Identifying the Unit

A second sticking point involves identifying the unit at various stages of the problem. When the problem starts, the unit is the whole brownie pan. We start with half of a whole pan of brownies. When asked to find 3/4 of 1/2 of the pan, the unit or whole that is partitioned is half of the pan. Together, Questions 3 and 4 focus student reasoning on what each of the fractions represent when the problem is enacted. Question 5 in Figure 3b ("How much of what you started with do you need?") leads students to articulate that they need 3/4 of 1/2 of the whole pan. A teacher could then respond, "If you need 3/4 of 1/2 of the pan, how could you show that in your picture?" At this point, most students realize that they need to partition 1/2 of the pan into four equal parts and identify (e.g., shade or mark) three of the parts. Question 6 in Figure 3c ("How much of the whole pan do you need?") aims to get students to articulate that they need 3/4 of 1/2 of the whole pan. It can be helpful at this point to prompt students to write near their picture "3/4 of 1/2 of a whole pan." In other words, it is helpful for student to express in words and writing that the brownie pan scenario represents "3/4 of 1/2 of a whole pan." This will eventually support Question 10 in Figure 3c, which asks, "What number sentence could you write to show what the problem is asking you to do mathematically?"

3.3 Sticking Point 3: Expressing the Solution

A third sticking point involves expressing the solution based on what the problem is asking. Questions 7 and 8 in Figure 3c are designed to direct attention to what the problem is asking—what part of a whole pan of brownies would you get if you bought 3/4 of 1/2 of a pan of brownies? Often students say the solution is 3/4. While it is true that 3/4 of 1/2 of the pan is shaded, the solution is expressed as the portion of the whole pan. This requires a second shifting of units from finding part of 1/2 of a pan back to finding part of a whole pan. The solution, 3/8, is the part of the whole pan of brownies bought. Question 7 ("How many pieces are in the whole pan?") directs students to determine how many 1/8 pieces are in the whole pan. Question 8 ("How many of those pieces do you end up needing?") focuses on how many 1/8 pieces are being bought. This leads to answering what part of the whole brownie pan Ms. Jones bought.

3.4 Sticking Point 4: Overgeneralizing Multiplication

As students become experienced with shifting across different units, attention can be directed to a fourth sticking point. Students often think that multiplication leads to a product that is larger than the numbers being multiplied. While this is the case with whole number multiplication, it is not true for all numbers, such as fractions. Question 9 ("Do you have more that you started with or less?" and "Why does this make sense?") prompts students to look at their brownie pan picture and consider that when they multiply by a fraction, the solution is less than the initial amount. This sets up an opportunity to discuss what is happening when multiplying fractions and why fraction $\times$ fraction multiplication leads to an answer with a smaller resulting value.

Finally, using Question 10 (”What number sentence would you write to show what the problem is asking you to do mathematically?”) prompts student to attach multiplication symbolism ($1/2 \times 3/4 = 3/8$) to the situation and their models. Connecting problem contexts, visual models, and symbolism supports understanding what fraction $\times$ fraction multiplication is. By using a context such as the brownie pans and asking students to develop visual models for problems, students engage in reasoning with fractional amounts and consider what is happening when taking a part of a part of a whole. As students respond to questions from the framework, they can see that a problem like $3/4 \times 1/2$ is not just about multiplying the numerators and multiplying the denominators to find a solution. Students become more familiar with fractions as quantities and gain experience with using fractions as scale factors.

4 Conclusion

NCTM (2014) argues for the importance of letting students engage in productive struggle. Oftentimes, teachers are concerned that if they do not demonstrate up front for students how to solve a problem, it will lead to confusion among students. Finding ways to support students as they work through difficult problems and avoiding telling them how to solve those problems is challenging. In addition, having a helpful problem context does not mean that students will not get stuck along the way. The sticking points described in this article are well documented in the research literature on learning to multiply with fractions (Mack, 2001; Armstrong & Bezuk, 1995). The questioning framework, used in conjunction with a context like the brownie pan problem, is a tool that teachers can use to support students as they work through these expected challenges.

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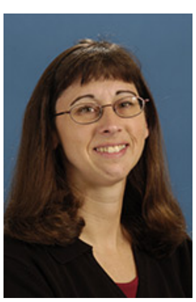

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