Establishing Equations and Encouraging Elves

Kendra Maree DeWater, *Miami University*

Abstract: *The author explores a problem about a team desiring to mount an attack on the North Pole. The problem offers interesting constraints that requires students to seek out patterns, set up scenarios, and find general forms. While working through the problem, the author explores ways in which problems of this sort foster student mathematical growth.*
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1 Introduction

Math teachers are always seeking exploratory tasks that engage students in interpreting, generalizing, and understanding mathematics in a contextualized situation. We look for problems that engage students from many backgrounds and learning styles, and help to develop critical and complex thinking. These rich tasks also are challenging, making the learning more meaningful and retainable (Day, 2015). In relation to this, I stumbled upon a problem about the North Pole that has direct connections to algebra, functions, and modeling.

I chose to format this paper in such a way that it seems as if a high school mathematics student is solving the problem on their own. It allows the teacher to see potential areas of confusion as well as interesting discussion pieces that may arise when presenting this problem to the class. This will be discussed further after the problem.

2 The Problem

This problem caught my attention because of its silly nature and because of its interesting implied process. It is stated as follows, "We are at the final base camp just 50 miles from the North Pole . . . Each of us can carry provisions that can keep us going for 20 miles . . . Not one of us, individually, could reach the pole, but if we work as a team in an efficient way, we can succeed" (Stevenson, 1992, p. 117). The problem suggests we can construct a general formula in which we can enter either the number of people in our expedition or the number of miles we need to go and get the other as an answer.

3 The Beginnings

I quickly realized that, to see how far someone could truly get in these circumstances, I did not want the person who makes it the farthest to return. Then, in each case, one person will not be returning to camp. If we only had one person in our expedition, they could travel 20 miles with their provisions. However, with two people, the situation would change. At mile five, each person would have 15 miles worth of provisions left. One person (denoted in blue in Figure 1) might give five miles worth of provisions to the purple person. The second person (denoted in purple in Figure 1) then has 20 miles of provisions, and the blue person has 10. The blue person cannot give any more to the purple person, so they go 2.5 more miles, and the first person then returns to camp. The purple person has 20 miles of provisions, and can thus make it to mile 25 before they die.

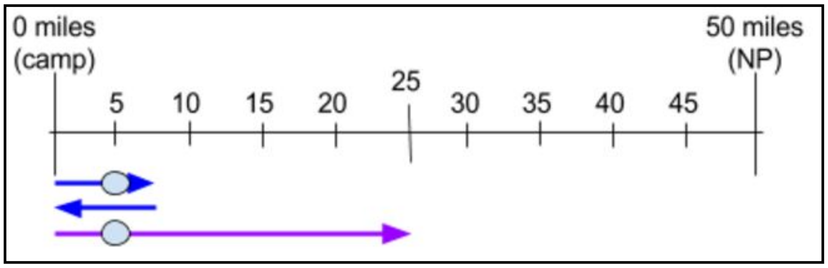

Fig. 1: *Distance with two people in expedition.*

Since focusing on exactly how many miles each person will have to travel would make it very difficult to generalize, I began to represent the situation with variables instead.

4 Exploring with Variables

4.1 Two People

Suppose the blue and purple person travel to some mile marker, a (see Figure 2). Then they will give up $20 - 2a$ miles of provisions to the purple person. This is because the blue person needs to travel back to camp, travelling a total of $2a$ of their possible 20 miles. The purple person would have already used a miles of provisions to get to mile a. Therefore, the purple person has $20 - a$ miles of provisions left. When they get the provisions from the blue person, they are adding $20 - 2a$ provisions to their current load. In total, then, the purple person has $20 - a + 20 - 2a$ provisions when they are at mile a.

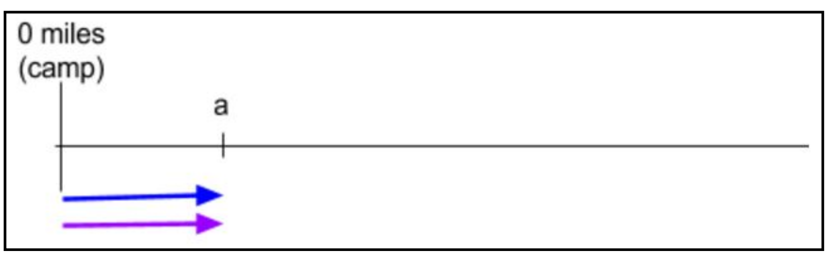

Fig. 2: *Distance with two people in expedition (general).*

The total number of provisions that the purple person has is $40 - 3a$. This number, though, can never exceed 20, due to the constraints of the problem.

$$40 - 3a \leq 20$$
$$a \geq 20/3$$

We know that the purple person has $40 - 3a$ provisions left, so they can go that many miles more. They have already gone a miles, so we know that the total number of miles they can go is

$40 - 3a + a = 40 - 2a$. If we want to maximize the number of miles that the purple person travels, we want to minimize the value of a. We discovered that the minimum value for a is 20/3. We get $40 - 2a = 40 - 2(20/3)$ which is equivalent to about 27 miles. This means that, if the blue person decides to turn back at the 20/3 mile mark, the purple person should be able to make it about 27 miles, and this is the maximum distance the purple person can go if they are only helped by one person. Sadly, they do not make it to the North Pole.

4.2 Three People

If person one goes a miles, they would give up the rest of their provisions as the blue person did in the previous problem, or $20 - 2a$ (see Figure 3). The second (purple) and third (green) people also traveled a miles, each having a remaining provision store of $20 - a$. This leaves, in total provisions for the two remaining people, $20 - 2a + 20 - a + 20 - a = 60 - 4a$. Of course, there can only be 40 provisions, since each person can carry a maximum of 20 provisions.

$$60 - 4a \leq 40$$
$$a \geq 5$$

Person one would now turn around. This leaves person two and three at mile a.

Fig. 3: *Distance with three people in expedition.*

Person two (purple) and person three (green) continue on to mile marker b (see Figure 4). We first need to realize that a miles is within b miles, and we do not want to double count what we have travelled. The purple person will turn around at this point, giving their $20 - 2b$ leftover provisions to the green person. Our total provisions remaining when person two (purple) leaves is $20 - 2a + 20 - 2b + 20 - b = 60 - 2a - 3b$.

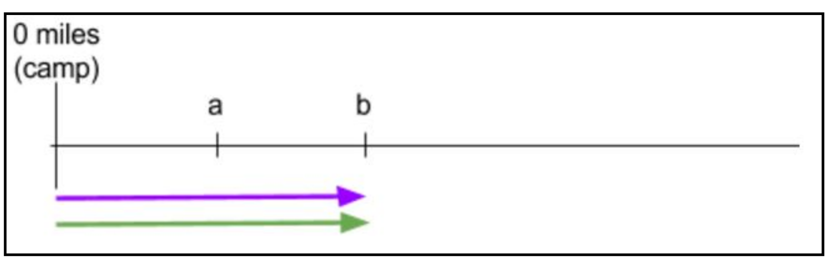

Fig. 4: *Distance with three people in expedition, part 2.*

Of course, for one person, this number cannot exceed 20 provisions.

$$60 - 2a - 3b \leq 20$$

We use the minimum value for a as we did before. When $a = 5$ we can see $b \geq 10$. Person 3 (green) has traveled b miles, and has $60 - 2a - 3b$ provisions remaining. Then the total miles travelled possible is $60 - 2a - 3b + b = 60 - 2a - 2b$. We want to maximize this value, so we minimize a and b. We found that the smallest a is five, and the smallest b is 10. The total number of miles travelled when using three people is $60 - 2(5) - 2(10) = 30$.

It seems as if, for n people, we can go $20n - 2(a + b + c + \ldots)$ miles where the inside of the parenthesis has $n - 1$ letters. To confirm this hunch, I explored the scenario with four people.

5 Four People

We follow the same process as before. When the first person leaves at a miles, there are $80 - 5a$ provisions left. There can only be 60 provisions left since we have three people.

$$80 - 5a \leq 60$$
$$a \geq 4$$

We know we will want to minimize this, so $a = 4$. When the second person turns around at mile b, there are $80 - 2a - 4b$ provisions remaining. This cannot exceed 40 provisions.

$$80 - 2a - 4b \leq 40$$

Remember that we said $a = 4$, so $b \geq 8$.

Again, we will want to minimize this, so $b = 8$. When the third person turns around at c miles, we have $80 - 2a - 2b - 3c$ provisions.

$$80 - 2a - 2b - 3c \leq 20$$

We said $a = 4$ and $b = 8$, so we know $3c \geq 36$.

To minimize, $c = 12$. The miles the fourth person can go is $80 - 2a - 2b - 3c + c = 80 - 2a - 2b - 2c = 20(4) - 2(a + b + c)$. Note that this looks like the general form that I had mentioned earlier. We said $a = 4$, $b = 8$, and $c = 12$. The total miles reached with four people is 32.

6 Generalizing

The pattern occurring is that the number of miles travelled with n people is $20n - 2(a + b + c + \ldots + (n - 1))$ people. We also see that, with the examples we tried, we could write a general form for a. In each case, the relationship we see is $20n - (n + 1)a \leq 20(n - 1)$.

$$20n - (n + 1)a \leq 20(n - 1)$$
$$a \geq 20/(n + 1)$$

We choose the minimum value for a in our problem, so we can say that $a = \frac{20}{n+1}$. This general form works for our values of a. If we look at b, we see a similar pattern emerge. We get $20n - 2a - bn \leq 20(n - 2)$. Remember, we said $a = \frac{20}{n+1}$, so this is really $20n - \frac{40}{n+1} - bn \leq 20(n - 2)$.

$$20n - \frac{40}{n + 1} - bn \leq 20(n - 2)$$
$$b \geq \frac{40}{n + 1}$$

We choose the minimum value again, so we say $b = \frac{40}{n+1}$. Let us check this with our value for b with $n = 3$. We obtain $b = 10$, which is what we got before! Now, clearly, a pattern is emerging with our values for a, b, c, etc. If this pattern continues, we should see that $c = \frac{60}{n+1}$. When we check this with our value for c when $n = 4$, we see that $c = 12$.

If these generalizations hold, it seems we have a general form. Since we have values for our a, b, c, etc., our general form is:

$$\begin{aligned} 20n - 2\left[\frac{20}{n+1} + \frac{40}{n+1} + \frac{60}{n+1} + \ldots + \frac{20(n-1)}{n+1}\right] &= 20n - 2\left[\frac{20}{n+1}\right](1+2+3+\ldots+(n-1)) \\ &= 20n - 2\left[\frac{20}{n+1}\right]\left[\frac{(n-1)(n)}{2}\right] \\ &= \frac{40n}{n+1} \text{ miles} \end{aligned}$$

Let us see if this form works for $n = 2$. We get $\frac{80}{3}$ which is about 27 miles, as we had! For $n = 3$, we get 30 miles, like we had. For $n = 4$, we get 32. This general form works for our examples.

7 Answering the Question

Now that we have our general form of $\frac{40n}{n+1}$ miles for n people, we can answer the initial question. If I want to get someone to the North Pole, all I need to do is plug in 50 for our miles and solve for n. Interestingly, when we solve for n, we get $n = -5$ as a solution. When I use GeoGebra to graph this general form as a function, the picture in Figure 5 is our general form.

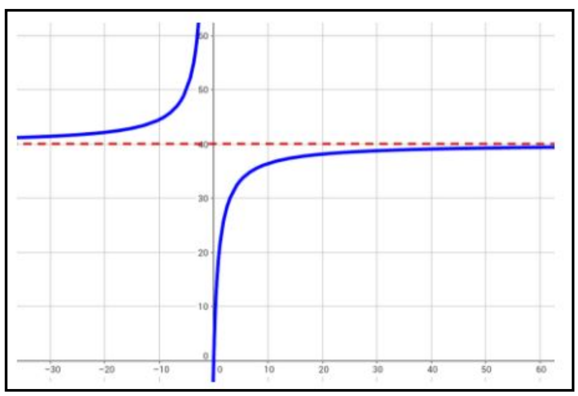

Fig. 5: *Graph of* $y = \frac{40x}{x+1}$.

There is a horizontal asymptote at 40 miles, regardless of the number of people we have in our expedition. If we wanted to go further than 40 miles, we would need to have a negative number of people in our expedition, which is clearly impossible. In other words, given the constraints of our situation, we are not able to reach the North Pole when it is 50 miles away. The farthest we can go, according to our graph in Figure 5, is just shy of 40 miles.

If we wanted to reach the North Pole, we would need our expedition members to carry more than 20 provisions. In fact, we can generalize our equation even further to work for any number of provisions. After working through a similar process as before, I developed the following general form $\frac{2pn}{n+1}$ miles, where p = miles of provisions carried by one person and n = number of people.

8 Conclusion

This exploration has been in the format of what a high school mathematics student might do when approaching this problem. Teachers know that encouraging exploration and allowing students to work through concepts on their own is incredibly important for the students mathematical development (Day, 2015). The problem is set in a context that high school students would find interesting; they could have a little fun with the problem itself. It is also challenging in terms of problem-solving and mathematical independence.

There are a few reasons that this problem is useful and interesting in a high school mathematics context. First, the level of mathematics is appropriate for high school students. It involves them working with inequalities, creating equations from a context, and, depending on how the teacher chooses to present the problem, modeling mathematics (Common Core Standards Initiative, 2010). The problem requires students to demonstrate a process of finding patterns and creating general forms; this is a necessary part of secondary mathematics and proofs.

Additionally, posing this problem to a high school mathematics classroom allows students to engage in a way they might not often have done in their past math classes. Particularly, students are doing mathematics from beginning to end; they interpret the problem into a mathematical context, explore and attempt various solutions, model their solutions to check for accuracy, and engage with each other to generalize and expand. The result is students teaching themselves and each other, and growing as mathematicians through doing so. This is especially true if the teacher chooses to have students present solutions and thought processes to the class, creating a truly collaborative experience.

Whether we are attacking the North Pole or trying to solve mathematics in a context in our daily lives, the skills to interpret, solve, and contextualize are incredibly important for our students and ourselves.

References

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Kendra Maree DeWater, `jozwiakm@miamioh.edu`, is a graduate student at in the Department of Teacher Education at Miami University in Oxford, Ohio. Her professional interests include the integration of social justice concepts into secondary mathematics curriculum. In her spare time, Kendra enjoys cooking, hiking, and spending time with her husband and puppy.